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A Note on the Characterization of Approval Voting on Dichotomous Preferences

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Abstract

This paper investigates approval voting axiomatically when the set of voters is fixed whereas the set of alternatives is assumed to vary. It is also assumed that each voter has a dichotomous preference over alternatives. Approval voting is then characterized by anonymity, neutrality, positive responsiveness, strategy-proofness, and stability on selected alternatives. This result sharpens the characterization theorem of Vorsatz (Approval voting on dichotomous preferences, Social Choice and Welfare, 28:127–141, 2007).

JEL classification: D71; D72

1 Introduction

Since the seminal work of Brams and Fishburn (1978), approval voting, in which each voter may vote for as many alternatives as he or she wishes, has been the subject of research for many authors. One direction is to axiomatically characterize the voting procedure, and a number of results including those reported by Fishburn (1978b), Sertel (1988), Alós-Ferrer (2006) and Xu (2010) have been presented.

This paper adds to the literature by presenting another characterization result for approval voting. More specifically, we characterize approval voting by anonymity, neutrality, positive responsiveness, strategy-proofness, and stability on selected alternatives. The first four properties are well known in social

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choice theory. Stability on selected alternatives, introduced by Arrow (1959), is equivalent to the combination of Sen's properties α and β. This property has been used, for example, by Vorsatz (2007), Alcalde-Unzu and Vorsatz (2013) and Sato (2014) for the characterization of approval voting. [1]

Our characterization is based on three specific assumptions. (a) Each voter has a dichotomous preference. A voter's preference over alternatives is said to be dichotomous if it has at most two indifference classes (often interpreted as the sets of "acceptable" and "non-acceptable" alternatives). With this assumption, we can analyze approval voting as a social choice function. (b) The set of voters is fixed, whereas (c) the actual set of alternatives can vary. Although most of the existing literature assumes that the set of voters is variable, many selection processes such as job interviews are performed with a fixed number of evaluators, and if there is an absentee among the evaluators, a replacement for him or her will be arranged. Assumption (b) supposes such a situation. However, by Assumption (c), we take account of possible situations such that some alternatives turn out to be infeasible before a vote. [2] This asymmetric treatment of voters and alternatives is justified shortly in relation to the result of Vorsatz (2007). It should also be stressed that this paper is the first to characterize approval voting under the three assumptions. [3]

Our characterization is closely related to a result of Vorsatz (2007). Theorem 1 of Vorsatz (2007) characterizes approval voting by anonymity, neutrality, positive responsiveness (its stronger variant, to be exact), strategy-proofness, stability on selected alternatives, and consistency in individuals. His setting is almost the same as ours except that he allows the actual set of voters to be a nonempty subset of the original set. In other words, he assumes that the set of voters can vary. Consistency in individuals then assumes that voters who are indifferent between two alternatives x and y do not affect the outcome if the feasible alternatives are only x and y. However, our result shows that for each fixed set of voters, approval voting can be characterized by the above first five properties. Therefore, our characterization theorem not only subsumes Theorem 1 of Vorsatz (2007) as a corollary but also implies that consistency in individuals is redundant in his theorem. Moreover, in Vorsatz (2007), it remains unanswered whether the stability of selected alternatives (or consistency in individuals) is independent from the other five properties. However,

[1] The term "stability on selected alternatives" is burrowed from Sato (2014).

[2] Stability of selected alternatives then restricts how the set of selected alternatives changes.

[3] For example, Fishburn (1978a) assumes (a) but neither (b) nor (c), whereas Baigent and Xu (1991) considers (b) and (c) but not (a). As mentioned below, Vorsatz (2007) deals with (a) and (c) but not (b). Finally, Alcalde-Unzu and Vorsatz (2013) and Sato (2014) take into account (c) but neither (a) nor (b).

we will see that whereas consistency in individuals is implied by the other five properties as mentioned above, stability on selected alternatives is independent from the others. This is another contribution of this paper.

The rest of this paper is organized as follows. In Section 2, we introduce notations and definitions. We state and prove our characterization theorem in Section 3.1. Section 3.2 confirms the independence of the properties used in the theorem. Concluding remarks are presented in Section 4. Finally, the Appendix provides the proofs of some lemmas.

2 Notations and Definitions

We follow the notations and definitions of Vorsatz (2007). Let K be a finite set of potential alternatives with the generic elements x, y, z. The cardinality of K is assumed to be greater than or equal to 3; i.e., $k \equiv |K| \geq 3$. We assume that the actual set of alternatives is drawn from the set $\mathcal{K} = \{S \in 2^K \setminus \{\emptyset\} : |S| \geq 2\}$. In other words, we consider a situation in which some alternatives in K may actually be infeasible.

Let $\mathcal{R}$ be the set of all reflexive, complete and transitive binary relations on $K \cup \{\emptyset\}$. For $R \in \mathcal{R}$, the asymmetric part and symmetric part of R are denoted by P and I, respectively. For $R \in \mathcal{R}$, define $G(R) = \{x \in K : xRy \text{ for all } y \in K \text{ and } xP\emptyset\}$ and $B(R) = \{x \in K : yRx \text{ for all } y \in K \text{ and } \emptyset Px\}$. A relation $R \in \mathcal{R}$ is said to be *dichotomous* if and only if $|G(R)| + |B(R)| = k$. The set of all dichotomous relations on $K \cup \{\emptyset\}$ is denoted by $\mathcal{D}$.

The set of n voters is denoted by $N = \{1, \cdots, n\}$ with $n = |N| \geq 2$. Each voter $i \in N$ has a preference over $K \cup \{\emptyset\}$ that is represented by a dichotomous relation $D_i \in \mathcal{D}$. A preference profile $D = (D_i)_{i \in N} \in \mathcal{D}^N$ specifies the dichotomous preferences of all voters. [4] For $D \in \mathcal{D}^N$ and $D'_i \in \mathcal{D}$, we denote the profile $(D_1, \cdots, D_{i-1}, D'_i, D_{i+1}, \cdots, D_n)$ by $(D'_i, D_{N \setminus \{i\}})$. For $D_i \in \mathcal{D}$, let

$$N(D; x, y) = \{i \in N : x \in G(D_i) \text{ or } y \in B(D_i)\} = \{i \in N : xR_i y\}$$

and

$$\bar{N}(D; x, y) = \{i \in N : x \in G(D_i) \text{ and } y \in B(D_i)\} = \{i \in N : xP_i y\}.$$

For $D \in \mathcal{D}^N$ and $x \in K$, define $N_x(D) = |\{i \in N : x \in G(D_i)\}|$, which is interpreted as the set of voters who "approve of" x.

[4] For notational simplicity, we denote preference profiles by letters in normal font rather than in bold font.

For $S \in \mathcal{K}$, the *social choice function* $f^S : \mathcal{D}^N \to 2^S \setminus \{\emptyset\}$ is a set-valued function that associates each profile $D \in \mathcal{D}^N$ with a nonempty subset $f^S(D)$ of S. The set $f^S(D)$ is interpreted as the set of alternatives selected from S when the preference profile is D. The *social choice rule* $\left\{f^S : \mathcal{D}^N \to 2^S \setminus \{\emptyset\}\right\}_{S\in\mathcal{K}}$ is a list of social choice functions for all possible $S \in \mathcal{K}$. [5]

Approval voting is the social choice rule that selects, for each S, all and only those alternatives with the greatest number of approvals.

Definition 2.1. The social choice rule $\left\{f^S : \mathcal{D} \to 2^S \setminus \{\emptyset\}\right\}_{S\in\mathcal{K}}$ is *approval voting* if for all $S \in \mathcal{K}$ and all $D \in \mathcal{D}^N$,

$$f^S(D) = \arg\max_{x\in S} \ N_x(D).$$

Note that if $N_x(D) = 0$ for all $x \in S$, approval voting selects all the alternatives in S.

Here we introduce five properties for the social choice rule.

Let σ be a permutation of N; i.e., σ is a bijection from N to N. Then, for $D \in \mathcal{D}^N$, define the profile $D^\sigma \in \mathcal{D}^N$ as $D^\sigma_i = D_{\sigma(i)}$ for all $i \in N$. Anonymity, which is a classical axiom in social choice theory, states that the names of voters do not affect the outcome.

Definition 2.2. The social choice rule $\left\{f^S : \mathcal{D}^N \to 2^S \setminus \{\emptyset\}\right\}_{S\in\mathcal{K}}$ is *anonymous* if for all $S \in \mathcal{K}$, all $D \in \mathcal{D}^N$ and all permutation σ of N,

$$f^S(D^\sigma) = f^S(D).$$

Let μ be a permutation of K. Then, for $D \in \mathcal{D}^N$, define the profile $D^\mu \in \mathcal{D}^N$ as $x \in G(D^\mu_i)$ if and only if $\mu^{-1}(x) \in G(D_i)$ for all $i \in N$ and $x \in K$. [6] Neutrality is another well-known axiom in social choice theory. It states that the names of alternatives do not affect the outcome.

Definition 2.3. The social choice rule $\left\{f^S : \mathcal{D}^N \to 2^S \setminus \{\emptyset\}\right\}_{S\in\mathcal{K}}$ is *neutral* if for all $S \in \mathcal{K}$, all $D \in \mathcal{D}^N$, and all permutation μ of K,

$$f^{\mu(S)}(D^\mu_N) = \mu\Big(f^S(D)\Big).$$

[5] It should be noted that in Vorsatz (2007), the term "social choice rule" is used to refer to the family of social choice functions $\left\{f^S\right\}_{S\in\mathcal{K}}$ that satisfies two specific conditions. See also Section 4.

[6] Since we use the symbols σ and μ only for the permutations on N and K, respectively, there should be no confusion between D^σ and D^μ.

The third property states that once x is selected between x and y (including a tie), if a voter who had preferred y to x now changes his or her mind to approve of x as well as y, then x should become the only selected alternative as far as everything else remains the same.

Definition 2.4. The social choice rule $\left\{f^S : \mathcal{D}^N \to 2^S \setminus \{\emptyset\}\right\}_{S \in \mathcal{K}}$ is *positively responsive* if it satisfies the following property. For $x, y \in K$, $i \in N$ and $D, D' \in \mathcal{D}^N$, suppose that $x \in B(D_i)$, $y \in G(D_i)$, $G(D'_i) = G(D_i) \cup \{x\}$ and $D_{N \setminus \{i\}} = D'_{N \setminus \{i\}}$. Then, $x \in f^{\{x,y\}}(D)$ implies $f^{\{x,y\}}(D') = \{x\}$.

Positive responsiveness is a weaker property than *strict monotonicity*, which is introduced in Vorsatz (2007). Formally, $\left\{f^S : \mathcal{D}^N \to 2^S \setminus \{\emptyset\}\right\}_{S \in \mathcal{K}}$ is strictly monotonic if it satisfies the following property.

> For $x, y \in K$, $i \in N$ and $D, D' \in \mathcal{D}^N$, suppose that $D_{N \setminus \{i\}} = D'_{N \setminus \{i\}}$ and either
>
> (a) $x, y \in B(D_i)$ and $G(D'_i) = G(D_i) \cup \{x\}$, or
>
> (b) $x, y \in G(D_i)$ and $B(D'_i) = B(D_i) \cup \{y\}$.
>
> Then, $x \in f^{\{x,y\}}(D)$ implies $f^{\{x,y\}}(D') = \{x\}$.

It is easily shown that strict monotonicity implies positive responsiveness. However, as the following example shows, the converse is not true. Fix $x, y \in K$, and define the social choice rule $\left\{f_0^S : \mathcal{D}^N \to 2^S \setminus \{\emptyset\}\right\}_{S \in \mathcal{K}}$ as follows. For $\{x, y\}$ and all $D \in \mathcal{D}^N$,

$$f_0^{\{x,y\}}(D) = \begin{cases} \{x, y\} & \text{if } N_x(D) = 0 \text{ and } N_y(D) = 0 \\ \{y\} & \text{otherwise;} \end{cases}$$

For all $S \neq \{x, y\}$ and all $D \in \mathcal{D}^N$,

$$f_0^S(D) = \arg\max_{z \in S} \; N_z(D).$$

It is easy to check that $\left\{f_0^S\right\}_{S \in \mathcal{K}}$ is positively responsive. However, it is not strictly monotonic. Let $K = \{x, y, z\}$ and $N = \{1, 2\}$. If $D \in \mathcal{D}^N$ is such that $G(D_1) = G(D_2) = \{z\}$, it holds that $f_0^{\{x,y\}}(D) = \{x, y\}$. Now consider $D'_1 \in \mathcal{D}$ such that $G(D'_1) = G(D_1) \cup \{x\} = \{x, z\}$. If $\left\{f_0^S\right\}_{S \in \mathcal{K}}$ satisfies strict monotonicity, it must be that $f_0^{\{x,y\}}(D'_1, D_2) = \{x\}$. However, since $N_x(D'_1, D_2) \neq 0$, it follows that $f_0^{\{x,y\}}(D'_1, D_2) = \{y\}$, which shows the violation of strict monotonicity.

For $i \in N$ and $D_i \in \mathcal{D}$, let $\succsim_{D_i}$ denote i's preference relation on $2^K \setminus \{\emptyset\}$ when i's preference on $K \cup \{\emptyset\}$ is D_i. We assume that $\succsim_{D_i}$ is reflexive, complete and transitive, and denote its asymmetric part by $\succ_{D_i}$. We also assume that $\succsim_{D_i}$ satisfies the following.

***Condition* P** $\{x\} \succ_{D_i} \{x, y\} \succ_{D_i} \{y\}$ if and only if $x \in G(D_i)$ and $y \in G(B_i)$.

***Condition* R** For all $S, T \in 2^K \setminus \{\emptyset\}$, if $S \subset G(D_i)$, $T \subset B(D_i)$, or $[S \setminus T \subset G(D_i)$ and $T \setminus S \subset G(B_i)]$, then $S \succsim_{D_i} T$.

For $S \in \mathcal{K}$, the social choice function f^S is said to be *manipulable* by voter $i \in N$ if there exist $D \in \mathcal{D}^N$ and $D'_i \in \mathcal{D}$ such that $f^S(D'_i, D_{N\setminus\{i\}}) \succ_{D_i} f^S(D)$. Now we introduce the fourth property.

Definition 2.5. The social choice rule $\left\{f^S : \mathcal{D}^N \to 2^S \setminus \{\emptyset\}\right\}_{S\in\mathcal{K}}$ is *strategy-proof* if for all $S \in \mathcal{K}$ and all $D \in \mathcal{D}^N$, the social choice function f^S is not manipulable by any voter.

The fifth property, introduced in Arrow (1959), states that there exists a certain relationship among the sets of selected alternatives $f^S(D)$ $(S \in \mathcal{K})$. In Vorsatz (2007), this property is included in the definition of the social choice rules.

Definition 2.6. The social choice rule $\left\{f^S : \mathcal{D}^N \to 2^S \setminus \{\emptyset\}\right\}_{S\in\mathcal{K}}$ is *stable on selected alternatives* if for all $S, T \in \mathcal{K}$ with $S \subset T$, and all $D \in \mathcal{D}^N$,

$$f^S(D) = f^T(D) \cap S$$

whenever $f^T(D) \cap S \neq \emptyset$.

3 Results

3.1 Characterization

In this section, we prove that approval voting is characterized by the five properties introduced in the previous section.

First we present two lemmas to be used for the characterization. The proofs are given in the Appendix.

Lemma 3.1. *Suppose that $\{f^S : \mathcal{D} \to 2^S \setminus \{\emptyset\}\}_{S\in\mathcal{K}}$ is anonymous, neutral, positively responsive, strategy-proof, and stable on selected alternatives.*

Let $x, y \in K$ and $D, D' \in \mathcal{D}^N$ be such that $\bar{N}(D; x, y) = \bar{N}(D'; x, y)$ and $\bar{N}(D; y, x) = \bar{N}(D'; y, x)$. Suppose further that

$$C \equiv \bar{N}(D; x, y) \cup \bar{N}(D; y, x) = \bar{N}(D'; x, y) \cup \bar{N}(D'; y, x) \neq \emptyset$$

and $f^{\{x,y\}}(D'_C, D_{N \setminus C}) = \{x\}$. Then,

$$f^{\{x,y\}}(D') = f^{\{x,y\}}(D'_C, D'_{N \setminus C}) = \{x\}.$$

Proof. See the Appendix. □

Lemma 3.2. *Let $\{f^S : \mathcal{D}^N \to 2^S \setminus \{\emptyset\}\}_{S \in \mathcal{K}}$ be a social choice rule that is anonymous, neutral, positively responsive, strategy-proof, and stable on selected alternatives. Then, for all $x, y \in K$ and $D, D' \in \mathcal{D}^N$, if $N(D; x, y) = N(D'; x, y)$ and $N(D; y, x) = N(D'; y, x)$, it holds that $f^{\{x,y\}}(D) = f^{\{x,y\}}(D')$.*

Proof. See the Appendix. □

Roughly speaking, Lemma 3.1 states that under certain circumstances, the voters who are indifferent between x and y do not affect the outcome of the selection between x and y. The lemma is used to prove Lemma 3.2. The statement of Lemma 3.2 is almost the same as that of Lemma 1 in Vorsatz (2007) except that the latter omits anonymity and positive responsiveness (strict monotonicity, to be precise) using an *a priori* consistency condition for $\{f^S\}_{S \in \mathcal{K}}$ in the proof. (See also Section 4.)

We are now in a position to state and prove our main result.

Theorem 3.1. *The social choice rule $\{f^S : \mathcal{D} \to 2^S \setminus \{\emptyset\}\}_{S \in \mathcal{K}}$ is Approval Voting if and only if it is anonymous, neutral, positively responsive, strategy-proof and stable on selected alternatives.*

Proof of Theorem 1. Once Lemma 3.2 is proved, Theorem 3.1 is shown in almost the same manner as Theorem 1 is shown in Vorsatz (2007). However, since we use positive responsiveness, which is weaker than strict monotonicity used in Vorsatz (2007), a minor modification of the proof is needed. More specifically, we alter (b) of the proof of Theorem 1 in Vorsatz (2007) in the following way.

(b) Suppose that there exists a preference profile $D \in \mathcal{D}^N$ such that $N_x(D) > N_y(D)$ and $f^{\{x,y\}}(D) = \{\{y\}, \{x, y\}\}$. Then, let $v \equiv N_x(D) - N_y(D)$, and let V be the set of v voters arbitrarily chosen from $\bar{N}(D; x, y)$; i.e., $V \subset \bar{N}(D; x, y)$ with $|V| = v$. For arbitrary $i \in V$, define $D'_i \in \mathcal{D}$ as

$G(D'_i) = G(D_i) \cup \{y\}$. Since $x \in G(D_i)$ and $y \in f^{\{x,y\}}(D)$, positive responsiveness implies

$$f^{\{x,y\}}(D'_i, D_{N\setminus\{i\}}) = \{y\}.$$

By inductively applying this argument to the remaining $i \in V$, we obtain $D'_V \in \mathcal{D}^V$ such that $x, y \in G(D'_i)$ for all $i \in V$, and

$$f^{\{x,y\}}(D'_V, D_{N\setminus V}) = \{y\}.$$

However, since $N_x(D'_V, D_{N\setminus V}) = N_y(D'_V, D_{N\setminus V})$ $(= N_y(D)+v)$, it follows from part (a) of the proof of Theorem 1 in Vorsatz (2007) that

$$f^{\{x,y\}}(D'_V, D_{N\setminus V}) = \{x, y\}.$$

This is a contradiction, and thus, $N_x(D) > N_y(D)$ implies $f^{\{x,y\}}(D) = \{x\}$.

The rest of the proof remains the same as that of Vorsatz (2007). □

3.2 Independence of the five properties

In this section, we show that each of the five properties in Theorem 3.1 is independent from the other four properties. [7]

3.2.1 Anonymity

Let q be the real-valued function defined on N such that $q(i) \neq q(j)$ for some $i, j \in N$. For $D \in \mathcal{D}^N$, let $\ell(x; D)$ be the real-valued function on K defined as

$$\ell(x; D) = \begin{cases} \sum_{i\in N: x\in G(D_i)} q(i) & \text{if } N_x(D) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

We then define the social choice rule $\left\{f_1^S : \mathcal{D}^N \to 2^S \setminus \{\emptyset\}\right\}_{S\in\mathcal{K}}$: for all $S \in \mathcal{K}$ and all $D \in \mathcal{D}^N$,

$$f_1^S(D) = \arg\max_{x\in S}\ \ell(x; D).$$

The rule $\left\{f_1^S\right\}_{S\in\mathcal{K}}$ is neutral, positively responsive, strategy-proof and stable on selected alternatives.

Let $K = \{x, y, z\}$. Define q as $q(1) = 1$, $q(2) = 2$ and $q(i) = 0$ for all $i \neq 1, 2$. If the preference profile $D \in \mathcal{D}^N$ is such that $G(D_1) = \{x\}$,

[7] The social choice rules used to prove the independence of anonymity and positive responsiveness are borrowed from Vorsatz (2007).

$G(D_2) = \{y\}$, it holds that $f^{\{x,y\}}(D) = \{y\}$. Consider the permutation σ of N defined as $\sigma(1) = 2$, $\sigma(2) = 1$ and $\sigma(i) = i$ for all $i \neq 1, 2$. Then, $G(D_1^\sigma) = \{y\}$ and $G(D_2^\sigma) = \{x\}$, and thus $f^{\{x,y\}}(D^\sigma) = \{x\} \neq f^{\{x,y\}}(D)$, which shows the violation of anonymity.

3.2.2 Neutrality

For a given $x \in K$, define the social choice rule $\left\{f_2^S : \mathcal{D}^N \to 2^S \setminus \{\emptyset\}\right\}_{S\in\mathcal{K}}$ as follows. For all $S \in \mathcal{K}$ and all $D \in \mathcal{D}^N$,

$$f_2^S(D) = \begin{cases} \{x\} & \text{if } x \in S \\ \arg\max_{y\in S} N_y(D) & \text{otherwise.} \end{cases}$$

The rule $\left\{f_2^S\right\}_{S\in\mathcal{K}}$ is anonymous, positively responsive, strategy-proof and stable on selected alternatives.

Let $K = \{x, y, z\}$. If the preference profile $D \in \mathcal{D}^N$ is such that $G(D_i) = \{x, y\}$ for all $i \in N$, it holds that $f^{\{x,y\}}(D) = \{x\}$. Consider the permutation μ of K defined as $\mu(x) = y$, $\mu(y) = x$ and $\mu(z) = z$. Then, $D^\mu = D$, and thus, $f^{\mu(\{x,y\})}(D^\mu) = f^{\{x,y\}}(D) = \{x\}$. However, $\mu\big(f^{\{x,y\}}(D)\big) = \{y\}$, which shows the violation of neutrality.

3.2.3 Positive responsiveness

Let $\left\{f_3^S : \mathcal{D}^N \to 2^S \setminus \{\emptyset\}\right\}_{S\in\mathcal{K}}$ be the social choice rule defined as follows. For all $S \in \mathcal{K}$ and all $D \in \mathcal{D}^N$,

$$f_3^S(D) = S.$$

It is obvious that $\left\{f_3^S\right\}_{S\in\mathcal{K}}$ satisfies anonymity, neutrality, strategy-proofness and stability on selected alternatives. It is also clear that $\left\{f_3^S\right\}_{S\in\mathcal{K}}$ violates positive responsiveness.

3.2.4 Strategy-proofness

For $D \in \mathcal{D}^N$, let $n(x; D)$ be the real-valued function on K defined as

$$n(x; D) = \begin{cases} \sum_{i\in N: x\in G(D_i)} \frac{1}{|G(D_i)|} & \text{if } N_x(D) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Using this function, we define the social choice rule $\left\{f_4^S : \mathcal{D}^N \to 2^S \setminus \{\emptyset\}\right\}_{S\in\mathcal{K}}$ as follows. For all $S \in \mathcal{K}$ and all $D \in \mathcal{D}^N$,

$$f_4^S(D) = \arg\max_{x\in S} \; n(x; D).$$

It is easily verified that $\left\{f_4^S\right\}_{S\in\mathcal{K}}$ satisfies anonymity, neutrality, positive responsiveness and stability on selected alternatives. However, as the following examples show, $\left\{f_4^S\right\}_{S\in\mathcal{K}}$ violates strategy-proofness.

Let $K = \{x, y, z\}$. If the preference profile $D \in \mathcal{D}^N$ is such that $G(D_1) = \{x, y\}$, $G(D_2) = \{z\}$ and $G(D_i) = \emptyset$ for all $i \neq 1, 2$, it holds that $f_4^K(D) = \{z\}$. Now consider $D_1' \in \mathcal{D}$ such that $G(D_1') = \{x\}$. Then, $f_4^K(D_1', D_{N\setminus\{1\}}) = \{x, z\}$. Under Condition **P**, this implies that voter 1 can manipulate f_4^K at D via D_1'.

3.2.5 Stability on selected alternatives

We define the social choice rule $\left\{f_5^S : \mathcal{D}^N \to 2^S \setminus \{\emptyset\}\right\}_{S\in\mathcal{K}}$ as follows. For all $S \in \mathcal{K}$ and all $D \in \mathcal{D}^N$,

$$f_5^S(D) = \begin{cases} K & \text{if } S = K, \\ \arg\max_{x\in S} N_x(D) & \text{otherwise.} \end{cases}$$

It is easily verified that $\left\{f_5^S\right\}_{S\in\mathcal{K}}$ satisfies anonymity, neutrality, positive responsiveness and strategy-proofness.

Let $K = \{x, y, z\}$. If the preference profile $D \in \mathcal{D}^N$ is such that $G(D_i) = \{x\}$ for all $i \in N$, it holds that $f_5^K(D) = K$. However, $f_5^{\{x,y\}}(D) = \{x\} \neq f_5^K(D) \cap \{x, y\} = \{x, y\}$, which shows the violation of stability on selected alternatives.

4 Concluding Remarks

We conclude this paper with two remarks on the relationship between our result and that of Vorsatz (2007).

First, Theorem 1 of Vorsatz (2007) can be derived as a corollary of our theorem. Theorem 1 of Vorsatz (2007) characterizes approval voting by anonymity, neutrality, strict monotonicity, strategy-proofness, stability on selected alternatives and consistency in individuals. The setting is almost the same as ours except that the possibility that the actual set of voters can be a nonempty subset of N rather than N itself is taken into account. Consistency in individuals then assumes that voters who are indifferent between x and y do not affect the outcome of the selection between the two alternatives. However, Theorem 1 of this paper shows that for each fixed set of voters, approval voting can be characterized by the above first five properties. This implies that Theorem 1 of Vorsatz (2007) follows from our theorem as a corollary, and moreover, consistency in individuals is redundant in Vorsatz's (2007) theorem.

Second, in Vorsatz (2007), stability on selected alternatives and consistency in individuals are included in the definition of the social choice rules, and their independence from the other properties is left unproven. However, whereas consistency in individuals is implied by the other five properties as mentioned above, a simple modification of the example provided in Section 3.2.5 shows that stability on selected alternatives is independent from the others (including consistency in individuals). This is another contribution of this paper.

Appendix

Proof of Lemma 3.1. Suppose that $f^{\{x,y\}}(D'_C, D'_{N\setminus C}) \in \{\{y\}, \{x,y\}\}$. The proof that follows is divided into three cases according to the relation between $|\bar{N}(D';x,y)|$ and $|\bar{N}(D';y,x)|$. In each case, a contradiction is obtained.

Case 1: $|\bar{N}(D';x,y)| = |\bar{N}(D';y,x)|$

In this case, $\bar{N}(D';x,y) \neq \emptyset$ and $\bar{N}(D';y,x) \neq \emptyset$ since $C \neq \emptyset$ by the supposition of the lemma. We construct a profile $D'' \in \mathcal{D}^N$ that satisfies three conditions:

(i) $f^{\{x,y\}}(D'') = \{x\}$,

(ii) $G(D''_i) = \{x\}$ for all $i \in \bar{N}(D'';x,y)$ and $G(D''_i) = \{y\}$ for all $i \in \bar{N}(D'';y,x)$,

(iii) $|\bar{N}(D'';x,y)| = |\bar{N}(D'';y,x)|$.

Let $i \in \bar{N}(D';x,y)$. If $G(D'_i) = \{x\}$, put $D''_i = D'_i$. It is then obvious that $f^{\{x,y\}}(D''_i, D'_{C\setminus\{i\}}, D_{N\setminus C}) = f^{\{x,y\}}(D'_C, D_{N\setminus C}) = \{x\}$.

Suppose that $E \equiv G(D'_i) \setminus \{x\} \neq \emptyset$. In this case, we define $D''_i \in \mathcal{D}$ as

$$G(D''_i) = G(D'_i) \setminus E \quad \text{and} \quad B(D''_i) = B(D'_i) \cup E.$$

Obviously, $G(D''_i) = \{x\}$. If $f^{\{x,y\}}(D''_i, D'_{C\setminus\{i\}}, D_{N\setminus C}) = \{\{y\}, \{x,y\}\}$, since $xP''_i y$, Condition **P** implies that the voter i can manipulate $f^{\{x,y\}}$ at

$(D''_i, D'_{C\setminus\{i\}}, D_{N\setminus C})$ via D'_i, which contradicts strategy-proofness. Therefore, $f^{\{x,y\}}(D''_i, D'_{C\setminus\{i\}}, D_{N\setminus C}) = \{x\}$.

By applying the above argument to the remaining $i \in \bar{N}(D'; x, y)$ one by one, we obtain $D''_{\bar{N}(D';x,y)} \in \mathcal{D}^{\bar{N}(D';x,y)}$ such that

$$f^{\{x,y\}}(D''_{\bar{N}(D';x,y)}, D'_{\bar{N}(D';y,x)}, D_{N\setminus C}) = \{x\}$$

and

$$G(D''_i) = \{x\} \quad \text{for all} \quad i \in \bar{N}(D'; x, y).$$

Next, choose arbitrary $i \in \bar{N}(D'; y, x)$. If $G(D_i) = \{y\}$, put $D''_i = D_i$. Then,

$$\begin{aligned} f^{\{x,y\}}(D''_{\bar{N}(D';x,y)}, &D''_i, D'_{\bar{N}(D';y,x)\setminus\{i\}}, D_{N\setminus C}) \\ &= f^{\{x,y\}}(D''_{\bar{N}(D';x,y)}, D'_{\bar{N}(D';y,x)}, D_{N\setminus C}) \\ &= \{x\}. \end{aligned}$$

Suppose that $F \equiv G(D'_i) \setminus \{y\} \neq \emptyset$. In this case, we define $D''_i \in \mathcal{D}$ as follows.

$$G(D''_i) = G(D'_i) \setminus F \quad \text{and} \quad B(D''_i) = B(D'_i) \cup F.$$

Since $i \in \bar{N}(D'; y, x)$ (i.e., yP'_ix), strategy-proofness together with Condition **P** implies

$$f^{\{x,y\}}(D''_{\bar{N}(D';x,y)}, D''_i, D'_{\bar{N}(D';y,x)\setminus\{i\}}, D_{N\setminus C}) = \{x\}.$$

By successively applying the above argument to the remaining $i \in \bar{N}(D'; y, x)$, and by putting $D''_i = D_i$ for all $i \in N \setminus C$, we obtain $D'' \in \mathcal{D}^N$ satisfying the condition (i) above. The profile D'' also satisfies (ii) and (iii) since $\bar{N}(D''; x, y) = \bar{N}(D'; x, y)$ and $\bar{N}(D'; y, x) = \bar{N}(D''; y, x)$.

A contradiction can now be obtained from this D''. Indeed, since $|\bar{N}(D''; x, y)| = |\bar{N}(D''; y, x)|$, there exists a permutation σ of N such that

$$\sigma(i) \in \begin{cases} \bar{N}(D''; y, x) & \text{if } i \in \bar{N}(D''; x, y) \\ \bar{N}(D''; x, y) & \text{if } i \in \bar{N}(D''; y, x) \\ \{i\} & \text{otherwise} \end{cases}$$

Anonymity then implies

$$f^{\{x,y\}}(D''^{\sigma}) = f^{\{x,y\}}(D''). \tag{1}$$

Let μ be the permutation of K defined as $\mu(x) = y$, $\mu(y) = x$ and $\mu(z) = z$ for all $z \neq x, y$. Then, by condition (ii) above, it is easily verified that $D''^{\mu} = D''^{\sigma}$. Finally, it follows from neutrality and (1) that

$$f^{\{x,y\}}(D'') = f^{\{x,y\}}(D''^{\sigma}) = f^{\{x,y\}}(D''^{\mu}) = \mu\Big(f^{\{x,y\}}(D'')\Big) = \{y\},$$

which is a contradiction.

Case 2: $|\bar{N}(D'; x, y)| < |\bar{N}(D'; y, x)|$

Let $v \equiv |\bar{N}(D'; y, x)| - |\bar{N}(D'; x, y)| > 0$, and let V be the set of v voters arbitrarily chosen from $\bar{N}(D'; y, x)$; i.e., $V \subset \bar{N}(D'; y, x)$ with $|V| = v$. For arbitrary $i \in V$, define $D''_i \in \mathcal{D}$ as

$$G(D''_i) = G(D'_i) \cup \{x\} \quad \text{and} \quad B(D''_i) = B(D'_i) \setminus \{x\}.$$

Since $y \in G(D'_i)$ and $f^{\{x,y\}}(D'_C, D_{N \setminus C}) = \{x\}$, positive responsiveness implies

$$f^{\{x,y\}}(D''_i, D'_{C \setminus \{i\}}, D_{N \setminus C}) = \{x\}.$$

By inductively applying this argument to the remaining $i \in V$, we obtain $D''_V \in \mathcal{D}^V$ such that $x, y \in G(D''_i)$ for all $i \in V$, and

$$f^{\{x,y\}}(D''_V, D'_{C \setminus V}, D_{N \setminus C}) = \{x\}.$$

Clearly,

$$\left| \bar{N}\Big((D''_V, D'_{C \setminus V}, D_{N \setminus C}); x, y\Big) \right| = \left| \bar{N}\Big((D''_V, D'_{C \setminus V}, D_{N \setminus C}); y, x\Big) \right|.$$

Then, if $C \setminus V = \emptyset$, neutrality implies

$$f^{\{x,y\}}(D''_V, D'_{C \setminus V}, D_{N \setminus C}) = \{x, y\},$$

which is a contradiction. However, if $C \setminus V \neq \emptyset$ and thus $\left| \bar{N}\Big((D''_V, D'_{C \setminus V}, D_{N \setminus C}); x, y\Big) \right| = \left| \bar{N}\Big((D''_V, D'_{C \setminus V}, D_{N \setminus C}); y, x\Big) \right| \geq 1$, we can deduce another contradiction by applying the same arguments as in Case 1 to the profile $(D''_V, D'_{C \setminus V}, D_{N \setminus C})$.

Case 3: $|\bar{N}(D'; x, y)| > |\bar{N}(D'; y, x)|$

Let $v \equiv |\bar{N}(D'; x, y)| - |\bar{N}(D'; y, x)| > 0$, and let V be the set of v voters arbitrarily chosen from $\bar{N}(D'; x, y)$; i.e., $V \subset \bar{N}(D'; x, y)$ with $|V| = v$. For arbitrary $i \in V$, define $D''_i \in \mathcal{D}$ as follows.

$$G(D''_i) = G(D'_i) \cup \{y\} \quad \text{and} \quad B(D''_i) = B(D'_i) \setminus \{y\}.$$

Since $x \in G(D'_i)$ and $y \in f^{\{x,y\}}(D'_C, D'_{N \setminus C})$, positive responsiveness implies

$$f^{\{x,y\}}(D''_i, D'_{C \setminus \{i\}}, D'_{N \setminus C}) = \{y\}.$$

By inductively applying this argument to the remaining $i \in V$, we obtain $D''_V \in \mathcal{D}^V$ such that $x, y \in G(D''_i)$ for all $i \in V$, and

$$f^{\{x,y\}}(D''_V, D'_{C\setminus V}, D'_{N\setminus C}) = \{y\}.$$

Note also that $|\bar{N}((D''_V, D'_{C\setminus V}, D'_{N\setminus C}); x, y)| = |\bar{N}((D''_V, D'_{C\setminus V}, D'_{N\setminus C}); y, x)|$. By neutrality, we may assume without loss of generality that $C \setminus V \neq \emptyset$, and thus $|\bar{N}((D''_V, D'_{C\setminus V}, D'_{N\setminus C}); x, y)| \geq 1$.

Then, in essentially the same manner as in Case 1, we can construct $D''' \in \mathcal{D}^N$ that satisfies the following conditions.

(1) $f^{\{x,y\}}(D''') = \{y\}$,

(2) $G(D'''_i) = \{x\}$ for all $i \in \bar{N}(D'''; x, y)$ and $G(D'''_i) = \{y\}$ for all $i \in \bar{N}(D'''; y, x)$,

(3) $|\bar{N}(D'''; x, y)| = |\bar{N}(D'''; y, x)|$.

However, in this case, it follows from anonymity and neutrality that $f^{\{x,y\}}(D''') = \{x, y\}$, which is a contradiction. $\square$

Proof of Lemma 3.2. The proof that follows is essentially the same as the proof of Lemma 1 in Vorsatz (2007) except that we rely on Lemma 3.1 whereas Vorsatz (2007) uses the property of *consistency in individuals*, which he includes in the definition of the social choice rules.

Suppose that there exist $x, y \in K$ and $D, D' \in \mathcal{D}^N$ such that $N(D; x, y) = N(D'; x, y)$ and $N(D; y, x) = N(D'; y, x)$, but $f^{\{x,y\}}(D) \neq f^{\{x,y\}}(D')$. Then, by neutrality, we may assume without loss of generality that $f^{\{x,y\}}(D) = \{x\}$ and $f^{\{x,y\}}(D') \in \{\{y\}, \{x, y\}\}$. Since $N(D; x, y) = N(D'; x, y)$ and $N(D; y, x) = N(D'; y, x)$, it is easily verified that

$$\bar{N}(D; x, y) = \bar{N}(D'; x, y) \quad \text{and} \quad \bar{N}(D; y, x) = \bar{N}(D'; y, x).$$

Put $C = \bar{N}(D; x, y) \cup \bar{N}(D; y, x) = \bar{N}(D'; x, y) \cup \bar{N}(D'; y, x)$. If $C = \emptyset$, neutrality implies

$$f^{\{x,y\}}(D) = \{x, y\},$$

which is a contradiction.

Next we prove that $f^{\{x,y\}}(D'_i, D_{N\setminus\{i\}}) = \{x\}$ for arbitrarily chosen $i \in C \neq \emptyset$. Suppose on the contrary that $f^{\{x,y\}}(D'_i, D_{N\setminus\{i\}}) \in \{\{y\}, \{x, y\}\}$. If $i \in \bar{N}(D; x, y)$ (i.e., xP_iy), voter i can manipulate $f^{\{x,y\}}$ at $(D'_i, D_{N\setminus\{i\}})$ via D_i. If $i \in \bar{N}(D; y, x)$ (i.e., yP_ix), voter i can manipulate $f^{\{x,y\}}$ at D via D'_i.

In either case, a contradiction is obtained. By replacing D_i with D'_i for the remaining $i \in C$ one by one, we can prove that $f^{\{x,y\}}(D'_C, D_{N\setminus C}) = \{x\}$.

Finally, it follows from Lemma 3.1 that $f^{\{x,y\}}(D') = f^{\{x,y\}}(D'_C, D'_{N\setminus C}) = \{x\}$, which contradicts the supposition of $f^{\{x,y\}}(D') \in \{\{y\}, \{x, y\}\}$. $\square$

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